Approximate Solution for the Viscous Boundary Layer on a Continuous Cylinder

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The boundary layer flow on a continuous flat surface issuing from a slot in a wall into an infinite still fluid was analyzed first by Sakiadis (1961a). His solution consisted of solving the well-known Blasius equation for two-dimensional boundary layer flow (for example, White, 1974) subject to different boundary conditions. The resulting solution is truly a similarity solution for the two-dimensional problem and provides the necessary solution for the early development of the boundary layer on a continuous cylinder upstream of any location where the transverse curvature becomes significant. Sakiadis (1961b) then solved the problem of boundary layer flow on a continuous cylinder issuing from a wall. This was a momentum integral solution incorporating a logarithmic velocity profile that was suggested first by Glauert and Lighthill (1954). Sakiadis modified the profile to account for different boundary conditions. Middleman and Vasudevan (1970) claimed to have obtained a similarity solution for this problem; however, as was pointed out by Fox and Hagen (1971), a similarity solution does not exist for this flow.

A common method for transforming the boundary layer equations written in cylindrical coordinates was suggested first by Seban and Bond (1951). A drawback of this transformation is the large streamwise variations of the velocity profiles and boundary layer thickness that result. Wanous and Sparrow (1965) used this transformation with great success in their perturbation series solution for laminar flow longitudinal to a stationary circular cylinder with surface mass transfer. From their solutions, the dimensionless boundary layer thickness varied by a multiplicative factor of 15 over the entire solution domain. This was not a drawback for their method, but it is for other approximate methods that seek to neglect the prior development of the boundary layer flow. Sparrow et al. (1970) incorporated the transformation used both by Gortler (1957) and Meksyn (1961) very successfully in the development of their multiple-equation boundary layer theory. Their choice of a suitable transformation was based on a desire to reduce streamwise variation in the solution, which accomplished the goal better than the previously mentioned transformation. Their three-equation local similarity model yielded solutions of high precision at the expense of some calculational complexities. The results presented here represent the extension of the theory of Sayles (1984) for the viscous boundary layer flow over semi-infinite cylinders.

Analysis

The governing boundary layer equations, written in cylindrical coordinates, are:

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0 \tag{1}$$

and

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2}\right] \tag{2}$$

The first term on the righthand side of Eq. 2 introduces the transverse curvature effect that precludes the possibility of obtaining a similarity solution for this boundary layer flow. The radial velocity component, u, can be expressed in terms of the axial velocity component, w, as

$$u = -\frac{1}{r}\int_a^r r\frac{\partial w}{\partial z}\,dr \tag{3}$$

A dimensionless axial velocity component can then be defined as:

$$f(\eta, s) = w/w_o$$

where w_o is the velocity of the translating cylinder. Also, dimensionless independent variables η and s are defined as:

$$\eta = y(w_o/\nu z)^{1/2}, \qquad s = \nu z/w_o a^2$$

where a is the cylinder radius, and $y = r - a$ is the radial distance above the cylindrical surface. The axial variation of the function f is accounted for with its dependence on s, and the variable η is the usual Blasius similarity variable for two-dimensional, flat-plate boundary layer flow.

Upon substituting Eq. 3 into Eq. 2 and incorporating the dimensionless variables defined above, Eq. 2 becomes:

$$f'' = \frac{-f'}{2(1+\eta/c)}\int_0^\eta f d\eta - \frac{f'}{(\eta+c)}\left[1 + \int_0^\eta \eta f d\eta\right] + s\left[f\frac{\partial f}{\partial s} - \frac{f'}{(\eta+c)}\frac{\partial}{\partial s}\int_0^\eta \eta f d\eta - \frac{f'}{(1+\eta/c)}\frac{\partial}{\partial s}\int_0^\eta f d\eta\right] \quad (4)$$

where $c = a(w_o/\nu z)^{1/2}$ is a dimensionless curvature parameter, and the primes refer to differentiation with respect to η. Every term in Eq. 4 has some explicit dependence on the curvature parameter c, and the prior development or history of the boundary layer flow is accounted for in the second group of terms in square brackets on the righthand side of the equation. The transformation suggested here appears to lead to a more complicated form of the equation to be solved than that resulting from the transformation of Seban and Bond (1951). It, however, is favorable for the following reasons:

- The effect of transverse curvature is clearly evident in the terms containing the parameter c.
- The Blasius two-dimensional boundary layer equation is easily extracted from Eq. 4 by letting the curvature parameter c approach infinity (simultaneously s approaches zero). This assures that the initial development of the boundary layer will be the same as that occurring on a two-dimensional plane surface.
- As will be shown, the use of this transformation leads to dimensionless velocity profiles and a boundary layer thickness whose streamwise variations are small.

The effect of neglecting the terms in Eq. 4 that accounts for the prior history of the flow is easy to assess and is the central result here. It should also be noted that Eq. 4 is applicable to either two-dimensional plane flows or flows with transverse curvature.

Results and Discussion

Equation 4 was solved subject to the following boundary conditions:

$$f(0, s) = 1.0 \quad \text{and} \quad f(\infty, s) = 0.0$$

and at $s = 0$, and for all values of η, the solution of Eq. 4 must reduce to that presented by Sakiadis (1961a) for two-dimensional plane flow.

The difference-differential technique developed initially by Hartree and Wormsley (1937) and further by Smith and Clutter (1963) was used to solve Eq. 4. The streamwise derivatives were evaluated using a three-point, backward-difference formula that allows for nonuniform streamwise intervals. Two upstream velocity profiles are needed to evaluate the derivative terms. The initial profiles were generated by neglecting the streamwise derivative terms in Eq. 4, since they are quite small for small values of the dimensionless axial coordinate, s. Once two initial profiles are generated and the streamwise derivative terms evaluated, Eq. 4 becomes an ordinary differential equation that can be solved to obtain a local velocity profile and skin friction.

Table 1. Skin Friction ($C_f Re_z^{1/2}$) Results for Flow on a Continuous Cylinder in Axial Motion

s	Sakiadis (1961b)	Difference-Differential	Local Similarity
0.0	0.8165	0.8875	0.8875
0.01	0.8771	0.9644	0.9778
0.05	0.9606	1.053	1.081
0.10	1.019	1.117	1.154
0.50	1.254	1.374	1.438
1.00	1.419	1.554	1.634
5.00	2.061	2.239	2.355
10.0	2.502	2.707	2.842
50.0	4.170	4.462	4.644
100.0	5.307	5.658	5.865
500.0	9.624	10.16	10.45
1,000.0	12.59	13.26	13.59

A trial-and-error procedure is required to obtain the value of $f'(0, s)$ at each axial location. Once obtained, the surface shear stress may be expressed in a dimensionless form as:

$$C_f Re_z^{1/2} = 2f'(0, s)$$

where

$$C_f = 2\tau_w/\rho w_o^2 \quad \text{and} \quad Re_z = w_o z/\nu$$

Equation 4 was solved subject to the boundary conditions stated above and the results are presented in Table 1 along with the momentum integral results of Sakiadus (1961b). The column labeled difference-differential is the solution that includes the streamwise derivative terms, while the column labeled local similarity is the solution that neglects these terms. The differences in the results are in the range of 2 to 5% over the entire solution domain. Presented in Figure 1 are difference-differential and local similarity velocity profiles for various axial locations. There is a lag in the development of the difference-differential velocity profiles compared with those predicted from

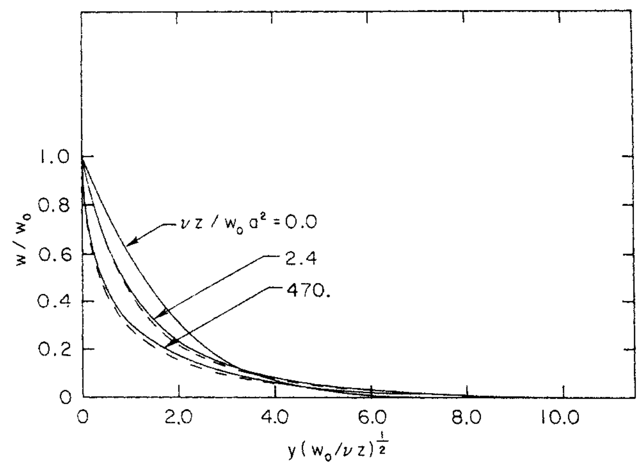

Figure 1. Velocity profiles at various axial locations.
-----, difference-differential: - - - -, local similarity

incorporating the local similarity assumption; the difference being the effect of the prior history of the flow on the shape of the profiles. It, however, is clear from Figure 1 that velocity profiles predicted from the local similarity assumption are accurate and the streamwise variation of the dimensionless boundary layer thickness is not very large. This solution technique shows great promise for the prediction of both viscous and thermal boundary layers on drawn fibers where the fiber mechanics has a strong influence on the development of the boundary layers.

Notation

a = radius of the cylinder
$c = a(w_o/\nu z)^{1/2}$, curvature parameter
$C_f = 2\tau_w/\rho w_o^2$, dimensionless wall shear stress
$f = w/w_o$, dimensionless axial velocity
r, z = radial and axial coordinates
$Re_z = w_o z/\nu$, Reynolds number
$s = \nu z/w_o a^2$, dimensionless axial coordinate
u, w = radial and axial velocity components
$y = r - a$, radial distance above the cylinder surface

Greek letters

$\eta = y(w_o/\nu z)^{1/2}$, Blasius similarity variable
ν = kinematic viscosity
ρ = density
τ = shear stress

Subscripts

$-1, -2$ = upstream axial locations where velocity profiles were generated
o = reference condition
w = conditions at the cylinder surface

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